JOURNAL OF COMPUTATIONAL PHYSICS 67, 214–222 (1986)

Simplified Second-Order Rezoning Algorithm for Generalized Two-Dimensional Meshes*,†

JOHN D. RAMSHAW

Theoretical Division, University of California, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

Received September 11, 1985; revised February 24, 1986

A second-order conservative rezoning algorithm for generalized two-dimensional meshes has recently been described by Dukowicz and Kodis (*SIAM J. Sci. Statist. Comput.*, in press). Here it is shown that this algorithm can be simplified by a different and more symmetrical choice of the flux vector **F** whose divergence is the cell density. The resulting formulation is the natural second-order generalization of the simplified first-order algorithm described earlier (*J. Comput. Phys.* **59**, 193 (1985)).

I. INTRODUCTION

Recently there has been a renewed interest in the general problem of transferring conserved quantities from one generalized finite-difference mesh to another [1–3]. This process is commonly referred to as rezoning or remapping; we shall use the former term. Dukowicz [1] described a rezoning method for two-dimensional meshes of arbitrary quadrilaterals in which the volumetric densities of the conserved quantities are considered uniform within each cell. The restriction to uniform cell densities implies that the discretization errors introduced by the rezoning are of first order in the spatial increments. The Dukowicz method is based on the introduction of a flux vector **F** corresponding to each conserved quantity Q of interest. The flux vector is defined in such a way that

$$\mathbf{V} \cdot \mathbf{F} = q, \tag{1}$$

where q is the volumetric density of the quantity Q (i.e., Q per unit volume) within the old (original) mesh. This relation was required to be satisfied everywhere within the region of interest, including points on old cell boundaries where q has finite discontinuities. It follows that the flux continuity condition

$$\mathbf{F}_1 \cdot \mathbf{n} = \mathbf{F}_2 \cdot \mathbf{n} \tag{2}$$

† Work performed under the auspices of the United States Department of Energy.

0021-9991/86 $3.00

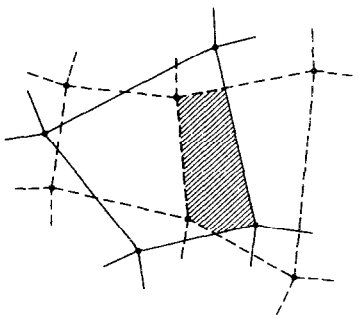

FIG. 1. Overlap areas created by superimposing the old mesh (solid lines) and the new mesh (dashed lines). Elementary overlap areas are those not further subdivided by lines of either mesh. A typical elementary overlap area is shaded.

must be satisfied on old cell boundaries, where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the limiting values of $\mathbf{F}$ on the boundary as approached from either side, and $\mathbf{n}$ is a vector normal to the boundary. (Equation (2) is easily derived by integrating Eq. (1) over a small "pillbox" on the boundary in the usual way.) Once $\mathbf{F}$ has been determined, the amount of Q in each cell of the new mesh is obtained by integrating Eq. (1) over the area of that cell and using the divergence theorem to reduce the area integral of $\nabla \cdot \mathbf{F}$ to a line integral of $\mathbf{F} \cdot \mathbf{n}$ over the new cell boundaries. This reduction represents a major simplification, which is the reason for introducing $\mathbf{F}$. Dukowicz devised an ingenious procedure for evaluating the resulting line integrals efficiently by systematically sweeping along the lines of the new mesh.

Of course, the flux vector $\mathbf{F}$ in the Dukowicz method is not uniquely determined by the old-mesh density field q. Dukowicz used the freedom available in the choice of $\mathbf{F}$ to impose the requirement that $\mathbf{F}$ have only one nonzero component in a curvilinear coordinate system aligned with the old mesh. This requirement facilitates the satisfaction of Eq. (2).

A simpler and more direct rezoning method for the case of uniform cell densities was subsequently developed [2]. This method was based on a simple algebraic formula for the elementary overlap areas created by superimposing the old and new meshes (see Fig. 1). In this approach there is no explicit use of a flux vector, but it was remarked that the method could be interpreted in terms of a flux vector $\mathbf{F}$ satisfying Eq. (1) in the *interior* of each old-mesh cell but not on the boundaries. The flux continuity condition of Eq. (2) is then no longer satisfied, and $\nabla \cdot \mathbf{F}$ consequently contains delta functions at the old cell boundaries. The area integral of $\nabla \cdot \mathbf{F}$ over a new cell then differs from the amount of Q in that cell by the contributions of these delta functions, but the latter can be evaluated and subtracted out. When this is done the result is the same as that obtained simply by consideration of the overlap areas. Alternatively, the delta functions at old cell boundaries can be circumvented by integrating $\nabla \cdot \mathbf{F}$ not over entire new cells at once but only over the individual elementary overlap areas. The amount of Q in each new

cell is then obtained simply by summing over the elementary overlap areas lying within it, and the same final result is again obtained.

Interpreting the method of Ref. [2] in terms of a discontinuous flux vector is not useful in the case of uniform cell densities, but it points the way toward useful higher-order generalizations of the Dukowicz method in which the flux continuity condition of Eq. (2) is relaxed. A second-order rezoning algorithm of this type has been described by Dukowicz and Kodis [3] (DK). Second-order accuracy is achieved by taking the density field q to be piecewise linear within each cell of the old mesh. A discontinuous flux vector $\mathbf{F}$ is utilized which satisfies Eq. (1) only in the interior of each old-mesh cell but not on its boundaries. Integration of $\nabla \cdot \mathbf{F}$ over each elementary overlap area then gives the amount of Q therein, regardless of the fact that Eq. (2) is not satisfied. These area integrals are converted to line integrals of $\mathbf{F} \cdot \mathbf{n}$ over the old and new cell boundaries, which are efficiently evaluated by systematically sweeping through both meshes in the manner of the first-order methods [1, 2].

The flux $\mathbf{F}$ in the DK method was again chosen to have only one nonzero component, in the spirit of the original Dukowicz method. In the absence of the flux continuity condition, however, such a choice is no longer particularly advantageous. Indeed, it has significant disadvantages: the resulting formulas acquire an artificial asymmetry between the two coordinate directions, and they are needlessly complicated. Our purpose here is to present a simpler and more symmetrical version of the DK method based upon a different choice of the flux vector $\mathbf{F}$. The present formulation is the natural second-order generalization of the simplified first-order method based on overlap areas [2], to which it reduces in the special case of uniform cell densities. As in the DK method, Cartesian and cylindrical coordinates are treated together within a unified framework. The resulting formulas subsume both cases and specialize easily to either.

The present formulation is simpler than that of DK, but the two are mathematically equivalent and will therefore produce identical results (to within roundoff errors). The simpler expressions of the present formulation will be more economical to evaluate numerically, but their evaluation is only a small part of the total computation so the savings will not be significant. Quite apart from questions of efficiency and aesthetics, however, the present formulation has the practical advantage that both the rezoning formulas themselves and any computer codes based on them are more transparent in structure and hence easier to work with.

The presentation is reasonably self-contained, but parts of it are somewhat abbreviated. These mostly concern aspects treated more fully in Refs. [1–3], with which the reader is assumed to be familiar.

II. The Flux Vector

Our first task is to determine a suitable flux vector of simple form for each cell in the old mesh, under the supposition that the density q varies linearly with position

within each such cell. In order to encompass both Cartesian and cylindrical coordinates within the same general framework, we replace Eq. (1) by

$$\nabla \cdot \mathbf{F} = Rq, \tag{3}$$

where $R = 1 - \alpha + \alpha x$, $\nabla = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y$, x and y are rectangular coordinates in the plane, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x- and y-directions, respectively, and $\mathbf{F}$ has been redefined to absorb a factor of R. The Cartesian case is realized by setting $\alpha = 0$, while $\alpha = 1$ for the cylindrical case. When $\alpha = 1$, x is the radial coordinate r and y is the axial coordinate z. Note that in both cases ∇ is defined as the gradient operator *in the plane*, all cylindrical effects being accounted for by the factors of R. The quantity Rq has the significance of Q per unit area in the xy-plane for a region having a depth of one length unit when $\alpha = 0$ and one radian when $\alpha = 1$. Combining the Cartesian and cylindrical cases in this way essentially reduces the latter to the former, so that both can be dealt with simply in terms of area integrals and the planar divergence theorem [3, 4].

Now focus attention on a particular but typical cell of the old mesh. The linear variation of q with position in this cell is expressed by

$$q = q_0 + \mathbf{G} \cdot (\mathbf{r} - \mathbf{r}_0), \tag{4}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is the position vector in the plane, $\mathbf{r}_0$ is the point within the cell at which the density has the value q_0, and $\mathbf{G}$ is the uniform value of ∇q within the cell. The values of q_0, $\mathbf{G}$, and $\mathbf{r}_0$ are of course constrained by the requirement that the area integral of Rq over the cell be equal to the known value of the total Q within it. It is convenient to let $\mathbf{r}_0$ be the centroid of the cell, whereupon q_0 becomes the average density of the cell [3]. The value of $\mathbf{G}$ is somewhat more difficult to specify, as it necessarily involves the values of q_0 in neighboring cells. Moreover, it is necessary to impose limits on the allowed values of $\mathbf{G}$ in order to prevent the rezoning from creating new extrema in the density field [3]. Of course, there is no unique procedure for determining and limiting $\mathbf{G}$, and various ways of doing so will not be explored here. For present purposes, we simply suppose that $\mathbf{G}$ has been determined for each cell of the old mesh by the prescriptions of DK [3], which appear to work well in practice.

Since $x = \mathbf{i} \cdot \mathbf{r}$, Eq. (4) and the definition of R may be combined with Eq. (3) to yield

$$\nabla \cdot \mathbf{F} = A + \mathbf{B} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{C} \cdot \mathbf{r}, \tag{5}$$

where

$$\begin{aligned} A &= (1 - \alpha)(q_0 - \mathbf{G} \cdot \mathbf{r}_0), \\ \mathbf{B} &= (1 - \alpha)\mathbf{G} + \alpha(q_0 - \mathbf{G} \cdot \mathbf{r}_0)\mathbf{i}, \\ \mathbf{C} &= \alpha \mathbf{i}\mathbf{G}. \end{aligned} \tag{6}$$

The form of Eq. (5) suggests that we attempt to determine a flux of the form

$$\mathbf{F} = aA\mathbf{r} + b(\mathbf{B}\cdot\mathbf{r})\mathbf{r} + c(\mathbf{r}\cdot\mathbf{C}\cdot\mathbf{r})\mathbf{r}, \tag{7}$$

the divergence of which is found to be precisely Eq. (5) if we take $a=\frac{1}{2}$, $b=\frac{1}{3}$, and $c=\frac{1}{4}$. We thereby obtain the remarkably simple and symmetrical flux vector

$$\mathbf{F} = \tfrac{1}{2}A\mathbf{r} + \tfrac{1}{3}(\mathbf{B}\cdot\mathbf{r})\mathbf{r} + \tfrac{1}{4}(\mathbf{r}\cdot\mathbf{C}\cdot\mathbf{r})\mathbf{r}, \tag{8}$$

in which A, $\mathbf{B}$, and $\mathbf{C}$ are given by Eq. (6). This is the flux vector upon which the present development is based. A similar flux vector is defined in each cell of the old mesh, the values of A, $\mathbf{B}$, and $\mathbf{C}$ therein being given by Eq. (6) in terms of the values of q_0, $\mathbf{G}$, and $\mathbf{r}_0$ for that cell.

III. Line Integrals

The basic ingredient in the rezoning procedure is the formula for the line integral of $\mathbf{F}\cdot\mathbf{n}$ along an arbitrary straight line segment. The line segments of interest are the sides of the elementary overlap areas, which are segments of the old and new mesh lines. Consider a typical straight line segment, with endpoints $\mathbf{r}_1$ and $\mathbf{r}_2$. (Which of the endpoints is considered point 1 is immaterial, but it is usually convenient to let it be the point with the lesser value of the spatial index that varies along the line.) The line segment will be considered as directed from point $\mathbf{r}_1$ to point $\mathbf{r}_2$, so that we may refer unambiguously to the areas lying to the left and right of it. These terms are defined with reference to an observer standing on the plane at point $\mathbf{r}_1$ and facing point $\mathbf{r}_2$. It is convenient to define

$$\begin{aligned} &\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1, \\ &L = (\Delta x^2 + \Delta y^2)^{1/2}, \\ &\mathbf{t} = L^{-1}(\mathbf{r}_2 - \mathbf{r}_1) = L^{-1}(\Delta x\mathbf{i} + \Delta y\mathbf{j}), \\ &\mathbf{n} = L^{-1}(\Delta y\mathbf{i} - \Delta x\mathbf{j}). \end{aligned} \tag{9}$$

Here L is just the length of the segment, $\mathbf{t}$ is the forward unit tangent, and $\mathbf{n}$ is the unit normal toward the right. Note that $\mathbf{t}\cdot\mathbf{n}=0$. The line integral we wish to evaluate is

$$I = \int_1^2 \mathbf{F}\cdot\mathbf{n}\, dl, \tag{10}$$

where dl is the element of length along the line segment. Points on the line segment are parameterized by l according to $\mathbf{r}=\mathbf{r}_1+l\mathbf{t}$, which varies from $\mathbf{r}_1$ at $l=0$ to $\mathbf{r}_2$ at $l=L$.

The fact that the flux $\mathbf{F}$ of Eq. (8) is proportional to $\mathbf{r}$ yields a considerable simplification, because $\mathbf{r}\cdot\mathbf{n}=\mathbf{r}_1\cdot\mathbf{n}$ is independent of l. Indeed, we easily find that

$$\mathbf{r}\cdot\mathbf{n}=L^{-1}(x_1y_2-x_2y_1), \tag{11}$$

which simply appears as a common factor outside the integral.

Inspection of Eq. (8) then shows that, apart from the trivial integral $\int dl=L$, the integrals we need to evaluate are

$$\mathbf{I}_1=\int_1^2 \mathbf{r}\,dl \qquad \text{and} \qquad \mathbf{I}_2=\int_1^2 \mathbf{rr}\,dl, \tag{12}$$

in terms of which I is simply given by

$$I=(x_1y_2-x_2y_1)\left(\frac{1}{2}A+\frac{1}{3L}\mathbf{B}\cdot\mathbf{I}_1+\frac{1}{4L}\mathbf{C}:\mathbf{I}_2\right). \tag{13}$$

The integrals $\mathbf{I}_1$ and $\mathbf{I}_2$ are easily evaluated, with the results

$$\begin{aligned}\mathbf{I}_1&=\tfrac{1}{2}L(\mathbf{r}_1+\mathbf{r}_2),\\ \mathbf{I}_2&=\tfrac{1}{3}L(\mathbf{r}_1\mathbf{r}_1+\mathbf{r}_2\mathbf{r}_2+\tfrac{1}{2}\mathbf{r}_1\mathbf{r}_2+\tfrac{1}{2}\mathbf{r}_2\mathbf{r}_1).\end{aligned} \tag{14}$$

Combining Eqs. (6), (13), and (14), we obtain

$$\begin{aligned}I=\tfrac{1}{2}(x_1y_2-x_2y_1)[&(1-\alpha+\tfrac{1}{3}\alpha x_1+\tfrac{1}{3}\alpha x_2)(q_0-\mathbf{G}\cdot\mathbf{r}_0)\\ &+\tfrac{1}{3}(1-\alpha+\tfrac{1}{2}\alpha x_1+\tfrac{1}{4}\alpha x_2)\,\mathbf{G}\cdot\mathbf{r}_1+\tfrac{1}{3}(1-\alpha+\tfrac{1}{2}\alpha x_2+\tfrac{1}{4}\alpha x_1)\,\mathbf{G}\cdot\mathbf{r}_2],\end{aligned} \tag{15}$$

which is our final formula for the basic line integral of interest. The corresponding quantity in the DK method [3] is $\Delta_1^2Q^k$, which is given by their Eq. (27). The present result is seen to be considerably simpler in structure.

The basic line integral I of Eq. (15) is of course a function of the segment endpoints $\mathbf{r}_1$ and $\mathbf{r}_2$, as well as the parameters q_0, $\mathbf{G}$, and $\mathbf{r}_0$ that characterize the density field of the old-mesh cell in question. In what follows, this functional dependence will be indicated by the notation $I(\mathbf{r}_1,\mathbf{r}_2;q_0,\mathbf{G},\mathbf{r}_0)$, which simply stands for the right member of Eq. (15).

IV. The Rezoning Procedure

Now consider a particular but typical elementary overlap area A (see Fig. 1). The amount of the quantity Q contained in A is just

$$Q_A=\int_A \nabla\cdot\mathbf{F}_A\,dx\,dy=\int_{\partial A}\mathbf{F}_A\cdot\mathbf{n}\,dl, \tag{16}$$

where $\mathbf{F}_A$ is the flux vector of the old-mesh cell in which A lies, ∂A is the perimeter of A, $\mathbf{n}$ is the outward unit normal, and the line integral is traversed in the counterclockwise direction. This line integral is just the sum of contributions of the form of Eq. (15) for each side s of the area A. Equation (16) therefore becomes

$$Q_A = \sum_s \varepsilon_s^A I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^A, \mathbf{G}^A, \mathbf{r}_0^A), \tag{17}$$

where the summation is over all the sides of A, and ε_s^A is either $+1$ or -1 according as A lies to the left or right, respectively, of side s. Here q_0^A, $\mathbf{G}^A$, and $\mathbf{r}_0^A$ are the values of q_0, $\mathbf{G}$, and $\mathbf{r}_0$ for the old-mesh cell in which A lies. The factor ε_s^A in Eq. (17) is necessary to allow for the case in which the line segment s is directed clockwise from the point of view of the area A. In this way the line segments s can be considered as separate entities independent of the particular overlap area under consideration, and their endpoints 1 and 2 can be labeled once and for all [2].

Equation (17) gives the contribution of the overlap area A to the total Q of the new-mesh cell in which A lies. The final value of Q in this new-mesh cell is just the sum of the Q_A contributions from all overlap areas within it. Division by the volume of this new-mesh cell then yields the final value of its average density q_0.

Just as in the first-order rezoning methods [1, 2], it is convenient to evaluate and sum these contributions by sweeping over all sides or segments s rather than over all overlap areas A. Each segment s is common to two overlap areas, the one on the left (L) and the one on the right (R), and therefore contributes to two Q_A's. Both of these contributions contain the same geometrical factors involving x_1^s, y_1^s, x_2^s, and y_2^s, so it is efficient to compute them at the same time. The way in which these contributions are calculated and used depends on whether side s is a segment of the old mesh or the new mesh.

If side s is a segment of the old mesh then it is common to two adjacent cells of the old mesh, the L and R cells. The L overlap area associated with side s lies entirely within the L cell, whose density field is characterized by q_0^L, $\mathbf{G}^L$, and $\mathbf{r}_0^L$. The R overlap area lies entirely within the R cell, whose density field is characterized by q_0^R, $\mathbf{G}^R$, and $\mathbf{r}_0^R$. Both overlap areas lie entirely within the same cell of the new mesh, and therefore the contributions of side s to both overlap Q's may be added to obtain the contribution of side s to Q in this cell. According to Eq. (17), the quantity Q in the new-mesh cell containing side s is therefore to be incremented by an amount

$$\Delta_s^0 = I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^L, \mathbf{G}^L, \mathbf{r}_0^L) - I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^R, \mathbf{G}^R, \mathbf{r}_0^R). \tag{18}$$

Of course, the detailed expression for Δ_s^0 that results when I is eliminated (see Eq. (15)) may be somewhat simplified by factoring out the common factors involving only $\mathbf{r}_1^s$ and $\mathbf{r}_2^s$.

If side s is a segment of the new mesh then it is common to two adjacent cells of the new mesh, the L and R cells. The L overlap area associated with side s lies

entirely within the L cell, while the R overlap area lies entirely within the R cell. Both overlap areas common to side s lie entirely within the same cell C of the old mesh, whose density field is characterized by q_0^C, $\mathbf{G}^C$, and $\mathbf{r}_0^C$. The contribution of side s to the L overlap area will then be an increment to Q_L, while that to the R overlap area will be an increment to Q_R. According to Eq. (17), the former contribution is just

$$\Delta_s^N = I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^C, \mathbf{G}^C, \mathbf{r}_0^C), \tag{19}$$

and the latter is just $-\Delta_s^N$. Therefore Q_L is to be incremented by Δ_s^N, while Q_R is to be decremented by the same amount.

One readily verifies that the above expressions for Δ_s^0 and Δ_s^N reduce to those of the simplified first-order method [2] when $\mathbf{G}$ is everywhere set to zero. The present formulation may therefore be regarded as the natural second-order generalization of the simplified first-order method.

The general procedure may therefore be summarized as follows. One first sweeps through all the lines of the old mesh, evaluating Δ_s^0 for each segment thereof by means of Eq. (18). The value of Q in the new-mesh cell containing segment s is to be incremented by Δ_s^0. Next one sweeps through all the lines of the new mesh, evaluating Δ_s^N for each segment thereof by means of Eq. (19). The value of Q in the new-mesh cell to the left of segment s is to be incremented by Δ_s^N, and the value of Q in the new-mesh cell to the right is to be decremented by Δ_s^N. The logic for sweeping along mesh lines and determining segment endpoints is identical to that of the first-order rezoning methods [1, 2]. Finally, after both mesh sweeps are complete, the resulting values of Q in the new-mesh cells are divided by the corresponding new-mesh cell volumes to obtain the average densities q_0 of the new-mesh cells.

Ambiguities arising from coincident old- and new-mesh segments may be resolved in a symmetrical manner analogous to that of the corresponding first-order method [2]. When such a segment is encountered while sweeping through the old mesh, Δ_s^0 is calculated in the usual way and Q in each of the two common new-mesh cells is incremented by $\frac{1}{2}\Delta_s^0$. When the segment is encountered again while sweeping through the new mesh, Δ_s^N is taken to be the average of $I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^L, \mathbf{G}^L, \mathbf{r}_0^L)$ and $I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^R, \mathbf{G}^R, \mathbf{r}_0^R)$, where the superscripts L and R refer to the two common old-mesh cells. One readily verifies that the net result of this symmetrization is that the segment in question contributes $I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^L, \mathbf{G}^L, \mathbf{r}_0^L)$ to the Q of the new-mesh cell on the left, and $-I(\mathbf{r}_1^s, \mathbf{r}_2^s; q_0^R, \mathbf{G}^R, \mathbf{r}_0^R)$ to the Q of the new-mesh cell on the right, and these are indeed the proper contributions.

ACKNOWLEDGMENT

I am grateful to J. K. Dukowicz for helpful comments and suggestions.

REFERENCES

1. J. K. DUKOWICZ, *J. Comput. Phys.* **54**, 411 (1984).
2. J. D. RAMSHAW, *J. Comput. Phys.* **59**, 193 (1985).
3. J. K. DUKOWICZ AND J. W. KODIS, Accurate conservative remapping (rezoning) for arbitrary Lagrangian–Eulerian computations, *SIAM J. Sci. Statist. Comput.*, in press.
4. J. D. RAMSHAW AND J. K. DUKOWICZ, "APACHE: A Generalized-Mesh Eulerian Computer Code for Multicomponent Chemically Reactive Fluid Flow," Los Alamos Scientific Laboratory Report LA-7427, Jan. 1979.